LOKA: JOURNAL OF ENVIRONMENTAL SCIENCES
P-ISSN - 2985-6531 (PRINT)
E-ISSN - 2986-3023 (ONLINE)
Indexed By : GARUDA, Google, ROAD, ISJD, OneSearch, EBSCO

Existence of Weak Solution for Black-Scholes Partial Differential Equation and Application of Energy Estimate Theorem in Sobolev Space

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Article Info:

Article History:
Received: 2024-06-24
Revised: 2024-07-01
Accepted: 2024-07-17

Keyword:
Weak solutions; Black-Scholes equations; Sobolev spaces; Option and Smooth functions

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Paper Type:
Research Paper

Abstract:

Purpose:
This paper aims to solve the BS second-order parabolic equation in Sobolev spaces to obtain weak solutions for financial applications, extending previous work in this field.

Methodology:
This paper constructs a set of functions that transforms the Black-Scholes partial differential equation into weak formulations. This study focuses on the Mathematics of finance, particularly the evolution of option pricing. An option's underlying asset involves agreements to buy or sell at a future strike price. The Black-Scholes (BS) equation, widely used in financial applications, models this.

Findings:
The analytical solutions, existence, uniqueness and other estimates were also obtained in weak form using boundary conditions to establish the effects of their financial implications in Sobolev spaces.

Implication:
The problem's regularity conditions were considered, and the coefficients and boundary of the domain are all smooth functions. To this end, this paper illustrates the definitions and assumptions that led to valuable assertions.

INTRODUCTION

Mathematical analysis in finance is critical not only to mathematicians or investors but also to the generality of the masses. So, understanding its dynamics will help economists, government, and opinion leaders adequately plan their investments effectively for future purposes. Hence, this study's particular area of interest is the Mathematics of finance, which deals with the evolution of option pricing. In other words, an underlying asset of an option is a business between parties who come together to agree on buying or selling an underlying asset at a particular strike price in the future for a fixed price. More so, the cost of the fundamental assets, which governs the growth of the option price, used the no-arbitrage argument to elucidate a Partial Differential Equation (PDE) concerning the expiration. The Black-Scholes (BS) equation has been used extensively in financial applications.

For instance, there is mammoth interest in financiers, mathematicians and statisticians over the partial differential equation Black and Scholes (1973) derived to analyze the European option on a stock market that does not pay dividends during the option's life. Mathematically, they restricted their analysis to conditions that made the problem simpler. The dynamics of option pricing describe the Black-Scholes PDE as a function of the security index and of the time to maturity of underlying assets. So, the recent development in option pricing has resulted in many diverse mathematical and computational methods, such as see etc. (Cont, 2006; Lindström et al., 2008; Fatone et al., 2008).

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LOKA Journal of Environmental Sciences

P-ISSN - 2985-6531 (PRINT)
E-ISSN - 2986-3023 (ONLINE)

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Therefore, Bianconi (2015) studied implied volatility and implied risk-free return rates in solving Black-Scholes equations systems. Their research established that options prices provide essential information for market participants for future expectations and market policies. Similarly, Babasola et al. (2008) analyzed the BS formula for evaluating European options; Hermite polynomials were applied. They concluded that the BS formula can be easily achieved without a partial differential equation. In another study of BS, Shin and Kim (2016) considered the BS terminal value problem and observed that their proposed method is better and more straightforward than the previous methods. In the work of Rodrigo and Mamon (2006), time-varying factors were incorporated in the explicit formula for different aspects of options to provide an exact solution for dividend-paying equity of options. Osu et al. (2009) applied the Crank-Nicolson numerical scheme to the BS model to consider the stability of the stock market price of the stochastic model. The results showed that stock prices were stable and that an increasing rate of stock shares was obtained. Osu (2010) investigated the variation of stock market prices using BS PDE for quite some time. The convergence to equilibrium of growth rate and sufficient conditions for stability was achieved. However, Nwobi et al. (2019) studied the Black-Scholes model because of its bias in mispricing options. They established a new technique of assessing pricing effects on the premise to reduce pricing bias. More financial models can be detailed in the works (Kasozi & Paulsen, 2005; Kasoji, 2022; Huang et al., 2017).

On the contrary, mathematics scholars have a similar interest in research based on topological and analytical foundations; such solutions exist but are not trivial. Hence, there are so many algebraic families with the above concepts: Hilbert spaces, Banach spaces and Sobolev spaces (Purnawan et al., 2024). Here, we consider Sobolev spaces because of their redefinition of differentiability, which starts with weak formulations to obtain weak solutions.

Nevertheless, a good number of scholars have used PDEs in Sobolev spaces for different reasons and results obtained in different ways such as DiPerna and Lions (1989) considered the existence, uniqueness and stability analysis for Ordinary Differential Equation (ODE) with coefficients in Sobolev spaces. Their results showed that renormalization solutions were used to analyze linear transport equations. Osu and Olunkwa (2014) viewed the solution to nonlinear BS equations and concluded that the bounded domain of weak solutions was extended to the entire domain via diagonal processes. Osu et al. (2020) considered a nonlinear Black Scholes Equation for incorporating transaction cost and portfolio risk as one of the financial models. These problems were solved in Sobolev space, and a weak solution with Fourier transformation properties was obtained.

This study aims to solve Black-Scholes' second-order parabolic equation in Sobolev spaces by obtaining weak solutions that can be used in financial applications. It extends the work of Osu and Olunkwa (2014) by considering BS PDE in such spaces. It is the first study to solve fully stochastic parabolic PDE with detailed proofs, definitions and assumptions in Sobolev spaces (Sanjaya et al., 2024).

The paper is arranged as follows: Section 2 presents the Mathematical preliminaries of Black-Scholes, Section 3 presents the Problem formulation of the Black-Scholes equation in Sobolev spaces, and Section 4 shows the main results of the Black-Scholes equation in Sobolev spaces. This paper concludes in Section 5.

METHODS

Sobolev Spaces. In a way, Sobolev spaces are analogous to spaces $C^{k,\alpha}\left(\overline{\varphi}\right)$ via $\varphi \subset \mathbb{R}^N$ an open set, which gives rise to usual differentiability being replaced by weak differentiability. Now, let $C_c^{\infty}\left(\varphi\right)$ be the set of infinitely differentiable functions via compact support $\phi : \varphi \to \mathbb{R}$. A function of this type $\phi \in C_c^{\infty}\left(\varphi\right)$ may be called a test function. $\phi \in C_c^{\infty}\left(\varphi\right)$ In each $u \in C'(\varphi)$ and taking note that every $\phi \in C_c^{\infty}\left(\varphi\right)$ has a compact support, Green's identity gives:

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$$\int_{\varphi} u\phi xi = -\int_{\varphi} uxi\phi \ \forall \phi \in C_c^{\infty}(\varphi),\ \forall i = 1,\ldots,N.$$

Definition 3.1: In each $1 \le p \le \infty$, the Sobolev space $W^{1,p}(\varphi)$ is defined as $g1,\ldots,gN \in L^p(\varphi)$ such that:

$$W^{1,p}(\varphi) = \left\{ u \in L^p(\varphi) / \int_{\varphi} u\phi xi = -\int_{\varphi} gi\phi \quad \forall \phi \in C_c^{\infty}(\varphi) \right\}$$

Where $uxi = gi$ the weak derivative $u \in W^{1,p}(\varphi)$ is unique, the gradient u is defined as $\nabla u = (ux_1,\ldots,ux_N)$ Assuming $p = 2 = 3$ we have $H^1(\varphi) = W^{1,2}(\varphi) = W^{1,3}(\varphi)$, respectively. The following lemma about L^p spaces is stated in order to prove that the weak derivative is unique:

Lemma 2.1: Let $\varphi \subset \mathbb{R}^N$ be an open set and let $u \in L^1_{loc}(\varphi)$ be such that $\int_{\varphi} u\phi = 0 \ \forall \phi \in C_c^{\infty}(\varphi)$. Then, $u = 0$ almost everywhere on φ.

Proposition 2.1: Let $u \in L^p(\varphi)$ be a function that has a weak derivative ux_i, so weak derivative is said to be unique except in a set of zero measure, that is to say, that $g, h \in L^p(\varphi)$ are functions such that $\int_{\varphi} u\phi x_i = -\int_{\varphi} g\phi = -\int_{\varphi} h\phi$ for every $\phi \in C_c^{\infty}(\varphi)$, then $g = h$ *a.e.*

Proof: assuming there exist $g, h \in L^p$ *such that* $\int_{\varphi} u\phi x_i = -\int_{\varphi} g\phi = -\int_{\varphi} h\phi, \forall \phi \in C_c^{\infty}(\varphi)$ $\psi = g - h,$ and $\psi = 0$ s. Hence, the weak derivative is well defined, i.e., up to zero measures.

Sobolev spaces can also be defined as a function $f \in L^p(\mathbb{R}^N)$ e_i the $i-th$ and vector of the canonical basis $\mathbb{R}^N$. The the $i-th$ partial derivative f exists in the L^p sense and equals $fx_i, if\ e^{-1}(\tau \in e_i f - f) \to -fx_i\ in\ L^p(\mathbb{R}^N)$ when $e \to 0$. the function $\tau \in e_i$ is defined $(\tau \in e_i f)(x) = f(x + \in e_i)$. Following the above definition, we also define the Sobolev space $W^{1,p}(\mathbb{R}^N) = \left\{ f \in L^p(\mathbb{R}^N) \mid fx_i \text{ exists in the } L^p \text{ sense for every } i = 1,\ldots,N \right\}$. The two definitions of the partial derivative fx_i can also be equal. In what follows, the first definition will be used instead of the second definition. For every $u \in W^{1,p}(\varphi)$, the norm u is defined as:

$$\| u \| W^{1.p} = \| u \|_p + \sum_{i=1}^{N} \| ux_1 \|_p$$

However, $H^1(\varphi)$ it is a Hilbert space equipped with scalar product as follows:

$$(u,v) H^1 = (u,v) L^2 + \sum_{i=1}^{N} (ux_i, vx_i) L^2$$

$H^1(\varphi)$ Is a Hilbert space (1.2) a norm induced in $H^1(\varphi)$ such that:

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$$\| u \|_{H^1}^{1} = \left(\| \mathrm{u} \|_2^2 + \sum_{i=1}^{N} \| u x_i \|_2^2 \right)^{\frac{1}{2}}$$

It is a norm which is equivalent to (1.1) for $p = 2$.

Theorem 2.2. (Energy estimates). There is a constant C, which depends majorly only on ϕ, T their coefficients are of L, such that.

$$\max_{0 \le t \le T} \| \mathrm{u}_m(t) \| L^2(\phi) + \| \mathrm{u}_m \| \mathrm{L}^2\left(0, T; H_0'(\phi)\right) + \| u_m' \| L^2\left(0, T, H^{-1}(\phi)\right)$$
$$\le C\left(\| f \| L^2(0, T; L^2(\phi)\right) + \| g \| L^2(\phi)),$$

For $m = 1, 2, \ldots,$. the proof, this theorem can be seen in the works (Osu & Olunkwa, 2014; Osu et al., 2020; Heston, 1993; Higham, 2004; Wilmott et al., 1995).

Mathematical Preliminaries of Black-Scholes Equation. In mathematical finance, a single asset for a contingent claim of the generic PDE is of the form:

$$\frac{\partial v}{\partial t} + a(x,t)\frac{\partial^2 v}{\partial x^2} + b(x,t)\frac{\partial v}{\partial x} + c(x,t)v = 0 \ .$$

Where t denotes time to maturity, x denotes the value of the underlying asset or functions of monotonic type (e.g., log (S); log-spot) and v denotes the value of the claim which is a function of x t and the following terms a (.), b (.) and c(.) are diffusion, convection and reaction Coefficients. (1.3)can also be written in the following manners :

$$\frac{\partial v}{\partial t} + a(x,t)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}\right) + b(x,t)\frac{\partial}{\partial x}\left(\beta(x,t)v\right) + c(x,t)v = 0 \ .$$

The above PDE describes the dynamics of the transition density of stochastic variables or quantities, for example, the stock value, which is seen in the Fokkper-Planoke (Kolmogorov forward) equation of probability measures. However, our interest in this paper is the parabolic financial PDE, which is governed by the dynamics of option pricing; hence, we have the following:

$$\frac{\partial v}{\partial t} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0.\ t > 0.$$

The details of the above option model can be found in the following books (Heston, 1993; Higham, 2004; Wilmott et al., 1995).

Weak Formulation of Black-Scholes Partial Differential Equation in Sobolev Spaces. Here, we investigate parabolic equations, which are Partial Differential Equations (PDEs). Let ϕ be an open, bounded subset of $\mathbb{R}^N$ setting $\phi_T = \phi \times (0,T]$ s at some specific fixed time $T > 0$.. Hence, the initial/boundary value problem is written as:

$$\begin{cases} u_t + Lu = f \ \ in\ \phi_T, \\ u = 0 \ \ on\ \partial\phi \times [0,T], \\ u = g \ \ on\ \phi \times \{t = 0\}. \end{cases}$$

Where $f : \phi_T \to \mathbb{R}$ and $g : \phi \to \mathbb{R}$ are given, and $u : \overline{\phi_T} \to \mathbb{R}$ is the unknown, $u = u(v,t)$. $L\ t$ a second-order partial differential operator is given for each time with the divergence form as follows.

$$Lv = -\sum_{ij=1}^{N} \left(\frac{1}{2}\sigma^2 s^2\right)^{ij} ((s,t)v_{si})s_j + \sum_{i=1}^{N} (rs)^{j} (s,t)v_{si} + r(s,t)v \ ,$$

Or also the non-divergence method.

$$Lu = -\sum_{ij=1}^{N} (v,t) u_{(v,t)i} (v,t) j + \sum_{ij=1}^{N} \left(\frac{1}{2}v\sigma^2\right)^{ij} (v,t) u_{(v,t)i} (v,t)\, j + \sum_{i=1}^{N} k\left[\theta - V_t\right]^{i} u_{(v.t)i} + r(v,t)u$$

With the coefficients $\left(\frac{1}{2}v\sigma^2\right)^{ij}, k\left[\theta - V_t\right]^i, r\left(ij = 1,\ldots,N\right)$.

Definition 1. 1: Partial differential operator $\frac{\partial}{\partial t} + L$ is said to exist with constant $\theta > 0$ and uniform parabolic properties such that,

$$\sum_{ij=1}^{N} \left(\frac{1}{2}v\sigma^2\right)^{ij} (v,t)\xi_i\xi_j \geq \theta \,|\, \xi \,|^2$$

For values of all $(v,t) \in \phi_T, \xi \in \mathbb{R}^N$.

Here is the usual example $\left(\frac{1}{2}v\sigma^2\right)^{1j} \equiv k\left(\theta - V_t\right)^i \equiv r \equiv f \equiv 0$ $L = -\Delta$: the PDE $\frac{\partial u}{\partial t} + Lu$ is the heat equation.

Weak solutions: Here, We shall consider the case w L ith the divergence form (1.8) and figure out an appropriate weak solution for the initial /boundary–value problem (1.4). The assumptions of weak solution are:

$$\left(\frac{1}{2}v\sigma^2\right)^{ij}, \mathrm{k}\left(\theta - V_t\right), r \in L^\infty\left(\phi_T\right), ij = 1,\ldots,\mathrm{N}\ .$$

$$f \in L^2\left(\phi_T\right)\ .$$

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P-ISSN - 2985-6531 (PRINT)
E-ISSN - 2986-3023 (ONLINE)
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$$g \in L^2(\phi) \ .$$

Assuming $\left(\frac{1}{2}v\sigma^2\right)^{ij} = \left(\frac{1}{2}v\sigma^2\right)^{ji}, ij = 1,\ldots,N$, introducing the concept of time-dependent bilinear form as:

$$B[u,v;t] := \int_\phi \sum_{ij=1}^{N} (v,t)^{ij} (.,t) u_{(v,t)i} v_{(v,t)j} + \int_\phi \sum_{ij=1}^{N} \left(\frac{1}{2}v\sigma^2\right)^{ij} (.,t) u_{(v,t)i} v_{(v,t)j}$$

$$+\sum_{i=1}^{N} k\left[\theta - V_t\right]^{i} (.,t) u_{(s,t)i} v + r(.,t) uvd(v,t).$$

For $u,v \in H_0'(\phi)$ and almost everywhere $(a.e)$. $0 \le t \le T$.

RESULTS AND DISCUSSION

Main Results of Black-Scholes equation in Sobolev spaces.

Definition 1. 2: Weak solution; in defining weak solution, we temporarily assume that $u = u(v,\mathrm{t})$ to be the parabolic problem of (1.7). Our point of focus via relating with u in a mapping such that:

$$u : [0,T] \to H_0'(\phi) \ ,$$

We define it as follows $\left[u(t)\right](v) := u(v,t), v \in \phi, 0 \le t \le T$. In the preceding, we shall consider u it not as a function of v and t together but as a mapping $u\ t$ of the space $H_0'(\phi)$ of function of v. Therefore, we make some clarifications in (1.4), and then we also define as well:

$$f : [0,T] \to L^2(\phi) \ ,$$

$$[f(t)](v) := f(v,t), v \in \phi, 0 \le t \le T \ .$$

Now, fixing a function $v \in H_0'(\phi)$ multiplying the PDE $\frac{\partial u}{\partial t} + Lu = f\ by\ v$ and integrating by parts to obtain as follows:

$$(u',v) + B[\mathrm{u},\mathrm{v};\mathrm{t}] = (f,v) \ .$$

In each $0 \le t \le T$, the pairing (,) denotes the inner product $L^2(\phi)$, and We notice that

$$u_t = g^0 + \sum_{j=1}^{N} g^j(v,t) j \ \ in\ \phi_T \ .$$

For $g^0 := f - \sum_{i=1}^{N} k\left[\theta - V_t\right]^i u_{(s,t)i} - ru \text{ and } g^j := \sum_{i=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij} u_{(v,t)} \;\; j = 1,\dots,N$.

As a result, the right-hand side of (1.16) lies in the Sobolev space $H^{-1}(\phi)$, with

$$\| u_t \| H^{-1}(\phi) \le \left(\sum_{j=0}^{N} \| g^j \|^2 \; L^2(\phi)\right)^{1/2} \le C\left(\| u \| H_0'(\phi) + \| \mathrm{f} \| \mathrm{L}^2(\phi)\right).$$

The estimate above suggests finding a weak solution with $u' \in H^{-1}(\phi)$ time $0 \le t \le T$; in any case, the first term in (1.15) is represented as $\langle u', v\rangle, \langle , \rangle$ the paring of $H^{-1}(\phi)$ *and* $H_0'(\phi)$.

Definition 1. 3: It suffices to say that a function $u \in L^2\left(0,T; H_0'(\phi)\right)$ with $u' \in L^2\left(0,T; H^{-1}(\phi)\right)$ is a weak solution of the parabolic initial/boundary-value problem (1.4) only if.

(i) $\langle u', v\rangle + B[\mathrm{u}, \mathrm{v}; \mathrm{t}] = (f, v)$, *for each* $v \in H_0'(\phi)$ *and a.e. time* $0 \le t \le T$.

(ii) $\mathrm{u}(0) = g$.

Definition 1.4: Existence of weak solutions, building a weak solution of the parabolic problem

$$\left.\begin{aligned} u_t + Lu &= f \;\; in \;\; \phi_T, \\ u &= 0 \;\; on \;\; \partial\phi \times [0,T], \\ u &= g \;\; on \;\; \phi \times \{t=0\}. \end{aligned}\right\}$$

First, we construct solutions of certain finite-dimensional approximations to (1.17) and then take the limits. It is known as Galerkin's method.

Setting a function w $w_k = w_k(v,t)$, k=1,…,as smooth such that,

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthogonal basis of } H_0'(\phi) .$$

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\phi) .$$

Taking $\{w_k\}_{k=1}^{\infty}$ the normalized Eigen-functions, which are appropriately complete, set for $L = -\Delta \;\, in \;\, H_0'(\phi)$. When considering a function $u_m : [o, \mathrm{T}] \to \mathrm{H}_0'(\phi)$, we fix a positive integer m.

$$u_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k \;\; ,$$

Which is where we pick the coefficients $d_m^k(t), 0 \le t \le T, k = 1,\dots,m,$ such that we have the following:

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$$d_m^k(0) = (g, w_k),\ k = 1,...,m\ .$$

$$(u_m', w_k) + B[u_m, w_k; t] = (f, w_k),\ 0 \le t \le T, k = 1,...,m$$

.

Where $(,)$ is the inner product $L^2(\phi)$? Seeking a function u_m of the form (1.20) that satisfies the projection (1.22) of the problem (1.17) onto the finite-dimensional subspace spanned through $\{w_k\}_{k=1}^m$.

Theorem 1.1: (Constructing of approximate solutions). A unique function u_m of the form (1.20) satisfies (1.21) and (1.22) each integer $m = 1, 2, \ldots$.

Proof: Let u_m be the mathematical structure (1.20). We remark first from (1.22) that,

$$(u_m'(t), w_k) = d_m^k(t)\ .$$

$$B[u_m, w_k; t] = \sum_{l=1}^{m} e^{kl}(t) d_m^l(t)\ .$$

For $e^{kl}(t) := B[\mathrm{wl}, \mathrm{w}_k; t]$, $\mathrm{k}, \mathrm{l} = 1, ..., \mathrm{m}$.

We write also,

$$f^k(t) := (f(t), w_k),\ k = 1,...,m\ .$$

Then (1.22) yields a linear system of ODE.

$$d_m^{k'}(t) + \sum_{l=1}^{m} e^{kl}(t) d_m^l(t) = f^k(t),\ k = 1,...,m\ .$$

With the following initial conditions of (1.21). Unique conditions exist according to the standard existence theory for ordinary differential equations (1.17). Hence, we want specific uniform estimates.

Theorem 1. 2. (Energy estimates). There is a constant C, which depends only on ϕ, T its coefficients L, such that.

$$\max_{0 \le t \le T} \| \mathrm{u}_m(t) \| L^2(\phi) + \| \mathrm{u}_m \| \mathrm{L}^2(0, T; H_0'(\phi)) + \| u_m' \| L^2(0, T, H^{-1}(\phi))$$
$$\le C(\| f \| L^2(0, T; L^2(\phi)) + \| g \| L^2(\phi)),$$

For $m = 1, 2, ...,$.

Proof:

Multiplying (1.19) by $d_m^k(t)$ and sum for $k = 1,...,m$ and then note (1.21) to obtain the quantities:

$$(u_m', u_m) + B[u_m, u_m; t] = (f, u_m)\ .$$

For a.e ., $0 \le t \le T$. Thus, we move to more to prove the existence of constants $\beta > 0, \gamma \ge 0$ such that the following condition holds:

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$$\beta \| u_m \|^2 \; H_0'(\phi) \leq B[u_m, u_m; t] + \gamma \| u_m \|^2 \; L^2(\phi) \quad .$$

$$0 \leq t \leq T, m = 1, \ldots \text{more so} \; |\left(f, u_m\right)| \leq \frac{1}{2} \| f \|^2 \; L^2(\phi) + \frac{1}{2} \| \mathrm{u}_m \|^2 \; L^2(\phi), \text{ and } \left(u_m', u_m\right)$$

for all,

$$= \frac{d}{dt}\left(\frac{1}{2} \| u_m \|^2 \; L^2(\phi)\right) \quad \textit{for a.e. } 0 \leq t \leq T.$$

Therefore (1.25) gives the inequality.

$$\frac{d}{dt}\left(\frac{1}{2} \| u_m \|^2 \; L^2(\phi)\right) + 2\beta \| u_m \| \; H_0'(\phi) \leq C_1 \| \mathrm{u}_m \|^2 \; L^2(\phi) + C_2 \| \mathrm{f} \|^2 \; L^2(\phi) \; .$$

For example. $0 \leq t \leq T$ where $C_1$and C_2 are constants? The following constants are defined:

$$\eta(t) := \| u_m(t) \|^2 \; L^2(\phi) \; ,$$

$$\xi(t) := \| \mathrm{f}(t) \|^2 \; L^2(\phi) \; .$$

So (1.30) means $\eta'(t) \leq C_1\eta(t) + C_2\xi(\mathrm{t})$ for a.e. , $0 \leq \mathrm{t} \leq \mathrm{T}$. the differential form of Gronwall's inequality, which gives the estimate.

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + \mathrm{C}_2 \int_0^t \xi(s) ds\right), 0 \leq t \leq T \; .$$

Since $\eta(0) = \| u_m(0) \|^2 \; L^2(\phi) \leq \| \mathrm{g} \|^2 \; L^2(\phi)$ (1.18) gives (1.28)- (1.30) the estimates as.

$$\max_{0 \leq t \leq T} \| \mathrm{u}_m(t) \|^2 \; L^2(\phi) \leq \mathrm{C}\left(\| \mathrm{g} \|^2 \; \mathrm{L}^2(\phi) + \| f \|^2 \; L^2(0, T; \mathrm{L}^2(\phi)\right).$$

From (1.30) and integrating from $0 \; to \; T$ and invoking the inequality above to obtain:

$$\| u_m \|^2 \; L^2\left(0, T; H_0'(\phi)\right) = \int_0^T \| u_m \|^2 \; H_0'(\phi) dt \leq C\left(\| g \|^2 \; L^2(\phi) + \| \mathrm{f} \|^2 \; L^2(0, T; L^2(\phi))\right).$$

For any $v \in H_0'(\phi), \; with \; \| v \| \; H_0'(\phi) \leq 1$ and write $v = v' + v^2$ where $v' \in span\{w_k\}_{k=1}^m$ and

$\left(v^2, w_k\right) = 0.k = 1, \ldots, m.$ since the function $\{w_k\}_{k=0}^\infty$s are orthogonal in

$H_0'(\phi), \| \mathrm{v}' \| \; \mathrm{H}_0'(\phi) \leq \| \mathrm{v} \| \; \mathrm{H}_0'(\phi) \leq 1.$ Applying (1.22), we reason for a.e. $0 \leq t < T$ that

$$\left(u_m', v'\right) + B[u_m, v'; t] = \left(f, v'\right),$$

Then (1.20) implies

$$\left\langle u_m', v\right\rangle = \left(u_m', v\right) = \left(u_m', v'\right) = \left(f, v'\right) - B[u_m, v'; t],$$

So

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$| \langle u'_m, v \rangle | \leq C\left(\| \mathrm{f} \| \mathrm{L}^2(\phi) + \| \mathrm{u}_m \| H'_0(\phi) \right),$

Since $\| v' \| H'_0(\phi) \leq 1,$

$\| u'_m \| H^{-1}(\phi) \leq C\left(\| f \| L^2(\phi) + \| \mathrm{u}_m \| H'_0(\phi) \right),$

Hence,

$$\int_0^T \| u'_m \|^2 \ H^{-1}(\phi)dt \leq C\int_0^T \| f \|^2 L^2(\phi) + \| u_m \|^2 \ H'_0(\phi)dt \leq C\left(\| g \|^2 \ L^2(\phi) + \| \mathrm{f} \|^2 \ L^2\left(0,T;L^2(\phi)\right)\right).$$

Existence and Uniqueness: We use limits $m \to \infty$ to build a weak solution to the initial boundary-value problem (1.17).

Theorem 1.3. (Existence of weak solution). Following (1.17), a weak solution exists.

Proof:

We want to prove the existence and uniqueness of the energy estimates (1.27), following the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2\left(0,\mathrm{T};\mathrm{H}'_0(\phi)\right)$ and $\{u'_m\}_{m=1}^{\infty}$ is bounded in $L^2\left(0,T;H^{-1}(\phi)\right)$. Thus, there exists a subsequence $\{u_{ml}\}_{l=1}^{\infty} \subset C\{u_m\}_{m=1}^{\infty}$ a function,

$$u \in L^2\left(0,T;H'_0(\phi)\right), \text{ with } u' \in L^2\left(0,T;H^{-1}{}_0(\phi)\right) \text{ such that we have (1.33).}$$

$$\begin{cases} u_{ml} \to u \text{ weakly in } L^2\left(0,T;H'_0(\phi)\right), \\ u'_{ml} \to u' \text{ weakly in } L^2\left(0,T;H^{-1}(\phi)\right). \end{cases}$$

Fixing an integer N and choosing a function $v \in C'\left([0,T];H'_0(\phi)\right)$ which gives the form:

$$v(t) = \sum_{k=1}^{N} d^k(t) w_k \ .$$

Where $\{d^k\}_{k=1}^{N}$ are given smooth functions? We choos $m \geq N$ e and multiply (1.22) by $d^k(t)$, sum $k = 1,\ldots,N$, and then by integration t concerning obtain:

$$\int_0^T \langle u'_m, v \rangle + B[u_m, v; t]dt = \int_0^T (f, v)dt \ .$$

Setting $m = ml$ and recall (1.31), to determine taking to weak limits such that,

$$\int_0^T \langle u', v \rangle + B[u, v; t]dt = \int_0^T (f, v)dt \ .$$

This type of equality holds for only functions $v \in L^2\left(0,T;H'_0(\phi)\right)$ as functions of the form (1.32) are dense in the space for functions of the form (1.32), in which we have:

$$\langle u', v \rangle + B[u, v; t] = (f, v) \ .$$

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For each $v \in H_0'(\phi)$ and a.e., $0 \le t \le T$. According to theorem 1.3, we also observe that $u \in C\left([0,T], L^2(\phi)\right)$.

$u(0) = g$ To prove this, we first note that from (1.34),

$$\int_0^T -\langle \mathrm{v}', \mathrm{u} \rangle + B[u,v;t]dt = \int_0^T (f,v)dt + \left(u(0), v(o)\right).$$

For each $v \in C'\left([0,T]; H_0'(\phi)\right)$ with $v(T) = 0$. Similarly, from (1.33), we deduce,

$$\int_0^T -\langle \mathrm{v}', \mathrm{u}_m \rangle + B[u_m,v;t]dt = \int_0^T (f,v)dt + \left(u_m(0), v(o)\right).$$

Setting $m = ml$ and once all over again, we invoke (1.34) to obtain,

$$\int_0^T -\langle \mathrm{v}', \mathrm{u}_m \rangle + B[u_m,v;t]dt = \int_0^T (f,v)dt + \left(\mathrm{g}, v(o)\right) .$$

Since $u_{ml(0)} \to g \; in \, L^2(\phi)$. As $v(0)$ is arbitrary, comparing (1.39) and (1.41), we can now say that $u(0) = g$.

Theorem 1. 4. (Uniqueness of weak solutions). There exists a weak solution of (1.17) using $f \equiv g \equiv 0$ is

$u \equiv 0$.

Proof:

In order to prove this point, we set $v = u$ in the identity of (1.38) $\left(\text{for} \equiv 0\right)$ we absorb, using theorem three such that,

$$\frac{d}{dt}\left(\frac{1}{2}\| u \|^2 \; L^2(\phi)\right) + B[\mathrm{u},\mathrm{u};\mathrm{t}] = \langle u', u \rangle + B[u,u;t] = 0 .$$

Since $B[u,u;t] \ge \beta \| u \|^2 \; H_0'(\phi) - \gamma \| \mathrm{u} \|^2 \; L^2(\phi) \ge -\gamma \| \mathrm{u} \|^2 \; L^2(\phi)$.

Using Gronwall's inequality and (1.43) implies (1.42). Hence, the details of Gronwall's inequality are seen in the following: (Evans, 2022; Imanol, 2016; Brezis & Brézis, 2011).

Regularity: Here, we describe the regularity of our weak solutions u to the Black-Scholes second-order parabolic equations. The primary objective is to prove that u it is a smooth function as far as the PDE coefficient; the boundary domain is smooth. The exciting thing about deriving the estimate is that it will increase some perceptions of how regularity assertions could be compelling temporarily $u = u(v,t)$. Let be a smooth solution to this initial- -value problem for the heat equation:

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$$\begin{cases} u_t - \Delta u = f \ in\ \mathbb{R}^N \times (0,\mathrm{T}], \\ u = g \ on\ \mathbb{R}^N \times \{t=0\}. \end{cases}$$

Assume also u tends to zero as $|v| \to \infty$ satisfactorily to explain the computations as follows: we calculate $0 \le t \le T$.

$$\int_{\mathbb{R}^N} f^2 dv = \int_{\mathbb{R}^N} (u_t - \Delta u)^2 dv - \int_{\mathbb{R}^N} u_t^2 - 2\Delta u u_t + (\Delta u)^2 = \int_{\mathbb{R}^N} u_t^2 + 2Du.Du_t + (\Delta u)^2\, dv .$$

$2Du.Du_t = \frac{d}{dt}\left(|\mathrm{Du}|^2\right)$ and consequently $\int_0^t \int_{\mathbb{R}^N} 2Du.Du_t dvds = \int_{\mathbb{R}^N} |\mathrm{Du}|^2 dv \,|_{s=0}^{s=t}$

$= \int_{\mathbb{R}^N} (\Delta u)^2 dv = \int_{\mathbb{R}^N} |\mathrm{D}^2\mathrm{u}|^2 dv.$

Applying the above two equalities in (1.44) and integrating time to obtain,

$$\sup_{0\le t\le T} \int_{\mathbb{R}^N} |Du|^2 dv + \int_0^T \int_{\mathbb{R}^N} u_t^2 + |D^2u|^2 \ dvdt \le C\left(\int_0^T \int_{\mathbb{R}^N} f^2 dvdt + \int_{\mathbb{R}^N} |\mathrm{Dg}|^2 \ dv\right).$$

We therefore estimate the $L^2 - norms\ of\ u_t$ and D^2u within $\mathbb{R}^N \times (0,T)$ in terms of the $L^2 -$ norms of f on $\mathbb{R}^N \times (0,T)$ and *the* $L^2 -$ norms of $Dg\ on\ \mathbb{R}^N$. We can now differentiate the PDE t concerning setting. $\bar{u} := u_t$

$$\begin{cases} \bar{u}_t - \Delta \bar{u} = \bar{f} \ in\ \mathbb{R}^N \times (0,\mathrm{T}], \\ \bar{u} = \bar{g} \ on\ \mathbb{R}^N \times \{t=0\}. \end{cases}$$

For $\bar{f} := f_t, \bar{g} := u_t(.,0) = f(.,0) + \Delta g$. Multiplying by $\bar{u}$, integrating by parts and invoking Gronwall's inequality, we infer that,

$$\sup_{0\le t\le T} \int_{\mathbb{R}^N} |u_t|^2 dv + \int_0^T \int_{\mathbb{R}^N} |Du_t|^2 dvdt \le C\left(\int_0^T \int_{\mathbb{R}^N} f^2 dvdt + \int_{\mathbb{R}^N} |\mathrm{D}^2\mathrm{g}|^2 + f(.,0)^2\, dv\right).$$

But then,

$$\max_{0\le t\le T} \| f(.,t) \| L^2(\mathbb{R}^N) \le C\left(\| f \| L^2(\mathbb{R}^N \times (0,T))\right) + \left(\| f_t \| L^2(\mathbb{R}^N \times (0,T))\right).$$

Following theorem 1.2 and writing $-\Delta u = + - u_i$ gives:

$$\int_{\mathbb{R}^N} |D^2u|^2 dv \le C \int_{\mathbb{R}^N} f^2 + u_t^2 dv .$$

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The Combinations of (1.48) - (1.50) leads us to the estimate of the below:

$$\underset{0\le t\le T}{Sup}\int_{\mathbb{R}^N}|u_t|^2dv+|D^2u|^2\ dv+\int_0^T\int_{\mathbb{R}^N}|Du_t|^2dvdt\le C\left(\int_0^T\int_{\mathbb{R}^N}f^2dvdt+\int_{\mathbb{R}^N}|\mathrm{D}^2\mathrm{g}|^2\ dv\right).$$

For some constant C .

Therefore, our earlier computations suggest estimates corresponding to (1.43) and (1.48) for weak solutions to the Black-Scholes' second-order parabolic equation.

Summary of solutions.

1)

$$u_t=g^0+\sum_{j=1}^N g^j(v,t)j\ \ in\ \phi_T\ .$$

For $g^0 := f-\sum_{i=1}^N k\left[\theta-V_t\right]^i u_{(s,t)i}-ru\ \ and\ \ g^j := \sum_{i=1}^N\left(\frac{1}{2}v\sigma^2\right)^{ij}u_{(v,t)}\ \ j=1,\dots,N$.

2)

$$u_t=g^0+\sum_{j=1}^N g^j(v,t)j\ \ in\ \phi_T\ .$$

For

$$g^0 :=\left(\lambda_1\lambda_2\right)^2 f-\sum_{i=1}^N k\left[\theta-V_t\right]^i u_{(s,t)i}\left(\lambda_1\lambda_2\right)^2-ru\left(\lambda_1\lambda_2\right)^2\ \ and\ \ g^j := \sum_{ij=1}^N\left(\frac{1}{2}v\sigma^2\right)^{ij}u_{(v,t)}\left(\lambda_1\lambda_2\right)^2\ \ j=1,\dots,N$$

3)

$$u_t=g^0+\sum_{j=1}^N g^j(v,t)j\ \ in\ \phi_T\ .$$

For

$$g^0 :=\left(\lambda_1+\lambda_2\right)^2 f-\sum_{i=1}^N k\left[\theta-V_t\right]^i u_{(s,t)i}\left(\lambda_1+\lambda_2\right)^2-ru\left(\lambda_1+\lambda_2\right)^2$$

$$and\ \ g^j := \sum_{i=1}^N\left(\frac{1}{2}v\sigma^2\right)^{ij}u_{(v,t)}\left(\lambda_1+\lambda_2\right)^2\ \ j=1,\dots,N$$

4)

$$u_t=g^0+\sum_{j=1}^N g^j(v,t)j\ \ in\ \phi_T\ .$$

For

$$g^0 :=\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2 f-\sum_{i=1}^N k\left[\theta-V_t\right]^i u_{(s,t)i}\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2-ru\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2$$

$$and\ \ g^j := \sum_{i=1}^N\left(\frac{1}{2}v\sigma^2\right)^{ij}u_{(v,t)}\left\{\left(\lambda_1+\lambda_2\right)^{-1}\right\}^2\ \ j=1,\dots,N$$

5)

$$u_t = g^0 + \sum_{j=1}^{N} g^j(v,t)j \ \ in\ \phi_T\ .$$

For

$$g^0 := \left\{(\lambda_1\lambda_2)^{-1}\right\}^2 f - \sum_{i=1}^{N} k\left[\theta - V_t\right]^i u_{(s,t)i}\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 - ru\left\{(\lambda_1\lambda_2)^{-1}\right\}^2$$

$$and\ \ g^j := \sum_{i=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij} u_{(v,t)}\left\{(\lambda_1\lambda_2)^{-1}\right\}^2 \ \ j = 1,\ldots,N$$

CONCLUSION

The analysis of Black-Scholes PDE in Sobolev spaces has been effortlessly established; hence, derivatives are well understood in a suitable weak sense to complete the space. From the analysis, a set of functions was constructed that transforms the Black-Scholes partial differential equation into weak formulations, which shows existence, uniqueness and other estimates in weak form using boundary conditions to establish the effects of its financial effects in Sobolev spaces. The problem's regularity conditions were considered, and the coefficients and the domain's boundary are all smooth functions.

Generally, the analytical methods of solving PDEs are non-trivial; they become complex when solved in Sobolev or other algebraic spaces. This difficulty worsens when the analysis of the problem is being sought. This paper aims to solve these Black-Scholes second-order parabolic equations in Sobolev spaces. The great challenge in analyzing this stochastic PDE is the definitions, assumptions, theorems and proofs, which are difficult to understand and apply appropriately.

However, in the following study, we shall examine the applications of these weak solutions and their implications for stock market price variations for capital investments.

Acknowledgment. We thank Almighty God for giving us the grace to create this vision. We also owe a debt of thanks to the journal's reviewers for their crucial comments.

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